Technische Universität Braunschweig

Model Order Reduction of Higher Order Systems

Joint work with Peter Benner and Philip Saltenberger

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Higher Order Linear Time-Invariant Systems

Higher Order Linear Time-Invariant Systems

Given matrices $P_j \in \mathbb{R}^{n\times n}, 0 \leqslant j \leqslant \ell$, $C_j \in \mathbb{R}^{p\times n}, 0 \leqslant j < \ell$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{p\times m}$ and an input function $u : [0, \infty) \to \mathbb{R}^m$, we seek the state function $x : [0, \infty) \to \mathbb{R}^m$ and the output function $y : [0, \infty) \to \mathbb{R}^p$ such that

$$P_\ell \frac{d^\ell}{dt^\ell} x(t) + P_{\ell-1} \frac{d^{\ell-1}}{dt^{\ell-1}} x(t) + \cdots + P_1 \frac{d}{dt} x(t) + P_0 x(t) = Bu(t)$$

$$Du(t) + C_{\ell-1} \frac{d^{\ell-1}}{dt^{\ell-1}} x(t) + \cdots + C_1 \frac{d}{dt} x(t) + C_0 x(t) = y(t)$$

with initial conditions

$$\left.\frac{d^j}{dt^j} x(t)\right|_{t=0} = x_0^{(j)}, \quad 0 \leqslant j \leqslant \ell,$$

where $x_0^{(j)} \in \mathbb{R}^n, 0 \leqslant j \leqslant \ell$ are given vectors.

Transfer Function

$$G(s) = D + \sum_{j=0}^{\ell-1} C_j (P_0 + sP_1 + s^2 P_2 + \cdots + s^\ell P_\ell)^{-1} B = D + \sum_{j=0}^{\ell-1} C_j (P(s))^{-1} B.$$

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Higher Order Linear Time-Invariant Systems

Model Order Reduction for Higher Order Linear Time-Invariant Systems

Given matrices

$$P_j \in \mathbb{R}^{n\times n}, 0 \leqslant j \leqslant \ell, C_j \in \mathbb{R}^{p\times n}, 0 \leqslant j < \ell, B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{p\times m}$$

and an input function $u : [0,\infty) \to \mathbb{R}^m$, we seek reduced order matrices

$$\hat{P}_j \in \mathbb{R}^{r\times r}, 0 \leqslant j \leqslant \ell, \hat{C}_j \in \mathbb{R}^{p\times r}, 0 \leqslant j < \ell, \hat{B} \in \mathbb{R}^{r\times m}, \hat{D} \in \mathbb{R}^{p\times m}$$

with $r \ll n$ such that

$$\hat{P}_\ell \frac{d^\ell}{dt^\ell}\hat{x}(t) + \hat{P}_{\ell-1}\frac{d^{\ell-1}}{dt^{\ell-1}}\hat{x}(t) + \cdots + \hat{P}_1\frac{d}{dt}\hat{x}(t) + \hat{P}_0\hat{x}(t) = \hat{B}u(t)$$
$$\hat{D}u(t) + \hat{C}_{\ell-1}\frac{d^{\ell-1}}{dt^{\ell-1}}\hat{x}(t) + \cdots + \hat{C}_1\frac{d}{dt}\hat{x}(t) + \hat{C}_0\hat{x}(t) = \hat{y}(t)$$

with suitable initial conditions yields a transfer function $\hat{G}(s)$ such that

$$\hat{G}(s) = G(s) + \mathcal{O}((s-s_0)^r) \text{ for some } s_0 \in \mathbb{C}.$$

Higher Order Linear Time-Invariant Systems

Galerkin Projection of Higher Order Linear Time-Invariant Systems

Given matrices $P_j \in \mathbb{R}^{n\times n}$, $C_j \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{p\times m}$, find a matrix $V \in \mathbb{R}^{n\times r}$ with orthonormal columns with $r \ll n$ and construct

$$\begin{aligned} \hat{P}_j &= V^T P_j V \in \mathbb{R}^{r\times r}, & \hat{B} &= V^T B \in \mathbb{R}^{r\times m}, \\ \hat{C}_j &= C_j V \in \mathbb{R}^{p\times r}, & \hat{D} &= D \in \mathbb{R}^{p\times m}, \end{aligned}$$

such that

$$\hat{G}(s) = G(s) + \mathcal{O}((s-s_0)^r) \ \text{ for some } s_0 \in \mathbb{C}.$$

Higher Order Linear Time-Invariant Systems

Standard approach: Linearization

Consider associated matrix polynomial $P(\lambda) = \lambda^\ell P_\ell + \lambda^{\ell-1} P_{\ell-1} + \cdots + \lambda P_1 + P_0 \in \Pi_\ell^n$ and convert it into $\lambda \mathcal{E} + \mathcal{A} \in \Pi_1^{\ell n}$ with the same eigenvalues.

Outline

- Illustrative examples
- Approach 1: MOR for higher order system by Freund (2005)
- (Approach 2: MOR for higher order system by Li, Bao, Lin, Wei (2011))
- New developments in linearization of matrix polynomials
  - Generalization of companion form linearization $\mathbb{L}_1$
  - Block Kronecker linearizations $\mathbb{G}_{r+1}$
- Higher order LTI systems and block Kronecker linearizations

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Illustrative examples

Gyroscopic system $P(\lambda) \in \Pi_2^n$

$$P(\lambda) = \lambda^2 M + \lambda G + K, \qquad M = M^T, G = -G^T, K = K^T, \quad M, G, K \in \mathbb{R}^{n\times n}.$$

Such problems arise, for example, in finite element discretization in structural analysis and in the elastic deformation of anisotropic materials. They are used to model vibrations of spinning structures such as the simulation of tire noise, helicopter rotor blades, or spin-stabilized satellites with appended solar panels or antennas.

Robot $P(\lambda) \in \Pi_4^n$

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0, \qquad P_i = (-1)^i P_i^T, \quad P_i \in \mathbb{R}^{n\times n}, i = 0, \dots, 4.$$

Such problems arise, e.g, from the model of a robot with electric motors in the joints.

T-even matrix polynomials

For both examples: $P(\lambda) = P(-\lambda)^T$.

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Back to Higher Order Linear Time-Invariant Systems

$$P_\ell \frac{d^\ell}{dt^\ell}x(t) + P_{\ell-1}\frac{d^{\ell-1}}{dt^{\ell-1}}x(t) + \cdots + P_1\frac{d}{dt}x(t) + P_0x(t) = Bu(t)$$

$$Du(t) + C_{\ell-1}\frac{d^{\ell-1}}{dt^{\ell-1}}x(t) + \cdots + C_1\frac{d}{dt}x(t) + C_0x(t) = y(t)$$

Let

$$z(t) = \begin{bmatrix} x(t) \\ \frac{d}{dt}x(t) \\ \vdots \\ \frac{d^{\ell-1}}{dt^{\ell-1}}x(t) \end{bmatrix}, \quad \mathcal{B}_F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix}, \quad \mathcal{A}_F = \begin{bmatrix} 0 & -I_n & 0 & \cdots & 0 \\ 0 & 0 & -I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -I_n \\ P_0 & P_1 & P_2 & \cdots & P_{\ell-1} \end{bmatrix},$$

$$\mathcal{E}_F = \begin{bmatrix} I_{(\ell-1)n} & \\ & P_\ell \end{bmatrix}, \quad \mathcal{C}_F = [C_0 \;\; C_1 \;\; \cdots \;\; C_{\ell-1}], \quad \mathcal{D}_F = D.$$

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Approach 1 [Freund 2005]

Approach 1: Linearization via the first companion form

The higher order system is equivalent to the first order system

$$\begin{aligned} \mathcal{E}_F \frac{d}{dt} z(t) + \mathcal{A}_F z(t) &= \mathcal{B}_F u(t) \\ y(t) &= \mathcal{D}_F u(t) + \mathcal{C}_F z(t) \\ z(0) &= z_0 \end{aligned}$$

where

$$z(t) = \begin{bmatrix} x(t) \\ \frac{d}{dt} x(t) \\ \vdots \\ \frac{d^{\ell-1}}{dt^{\ell-1}} x(t) \end{bmatrix}, z_0 = \begin{bmatrix} x_0^{(0)} \\ x_0^{(1)} \\ \vdots \\ x_0^{(\ell-1)} \end{bmatrix}, \mathcal{B}_F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B \end{bmatrix}, \mathcal{A}_F = \begin{bmatrix} 0 & -I_n & 0 & \cdots & 0 \\ 0 & 0 & -I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -I_n \\ P_0 & P_1 & P_2 & \cdots & P_{\ell-1} \end{bmatrix},$$

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- Transfer function
  $G(s) = \mathcal{D}_F + \mathcal{C}_F(s\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{B}_F = D + \sum_{j=0}^{\ell-1} C_j(P(s))^{-1}B \in \mathbb{C}[s]^{p\times m}$.
- $\mathcal{E}_F, \mathcal{A}_F \in \mathbb{R}^{\ell n \times \ell n}, \mathcal{B}_F \in \mathbb{R}^{\ell n \times m}$ are large and (block-) sparse.
- $\lambda\mathcal{E}_F + \mathcal{A}_F$ does not inherit any structure from $P(\lambda)$,
  that is, e.g., $P(\lambda) = P(\lambda)^T$ does not imply that $(\lambda\mathcal{E}_F + \mathcal{A}_F)^T = \lambda\mathcal{E}_F + \mathcal{A}_F$.

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- Rewrite $G(s) = \mathcal{D}_F + \mathcal{C}_F(s\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{B}_F$ for $s_0 \in \mathbb{C}$ such that $s_0\mathcal{E}_F + \mathcal{A}_F$ is nonsingular as

$$G(s) = \mathcal{D}_F + \mathcal{C}_F(I + (s - s_0)\mathcal{M}_F)^{-1}\mathcal{R}_F$$

with

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[Freund 2005]

- Rewrite $G(s) = \mathcal{D}_F + \mathcal{C}_F(s\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{B}_F$ for $s_0 \in \mathbb{C}$ such that $s_0\mathcal{E}_F + \mathcal{A}_F$ is nonsingular as

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Approach 1 [Freund 2005]

The matrices $\mathcal{M}_F$ and $\mathcal{R}_F$ have a particular structure

$$\mathcal{M}_F = (s_0\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{E}_F = (c \otimes I_n)\begin{bmatrix} M^{(1)} & M^{(2)} & M^{(3)} & \cdots & M^{(\ell)} \end{bmatrix} + \Sigma \otimes I_n,$$
$$\mathcal{R}_F = (s_0\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{B}_F = c \otimes R,$$

where

$$M^{(i)} = (P(s_0))^{-1}\sum_{j=0}^{\ell-i} s_0^j P_{i+j} \in \mathbb{C}^{n\times n}, i = 1, \ldots, \ell$$

$$R = (P(s_0))^{-1}B \in \mathbb{C}^{n\times m},$$

$$c = \begin{bmatrix} 1 \\ s_0 \\ s_0^2 \\ \vdots \\ s_0^{\ell-1} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & & \vdots \\ s_0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_0^{\ell-2} & \cdots & s_0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{\ell\times\ell}.$$

Approach 1

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Theorem (Freund 2005)

Let $\mathcal{M}_F = (c \otimes I_n)\begin{bmatrix} M^{(1)} & M^{(2)} & M^{(3)} & \cdots & M^{(\ell)} \end{bmatrix} + \Sigma \otimes I_n$, *and* $\mathcal{R}_F = c \otimes R$ *with* $c \in \mathbb{C}^{\ell}$, $c_j \neq 0, j = 1, \ldots, \ell$, $R \in \mathbb{C}^{n \times m}$, $M^{(i)} \in \mathbb{C}^{n \times n}$, $i = 1, \ldots, \ell$, $\Sigma \in \mathbb{C}^{\ell \times \ell}$. *Let* $\mathcal{W} \in \mathbb{C}^{\ell n \times r}$ *be any basis of the block-Krylov subspace* $\mathcal{K}_s(\mathcal{M}_F, \mathcal{R}_F)$, $r \leqslant sm$. *Then* $\mathcal{W}$ *can be represented in the form*

$$\begin{bmatrix} WU^{(1)} \\ WU^{(2)} \\ \vdots \\ WU^{(\ell)} \end{bmatrix}$$

where $W \in \mathbb{C}^{n \times r}$ *and, for each* $i = 1, 2, \ldots, \ell$, $U^{(i)} \in \mathbb{C}^{r \times r}$ *is nonsingular and upper triangular.*

- $\mathcal{K}_s(\mathcal{M}_F, \mathcal{R}_F) \subset \mathbb{C}^{\ell n}$ consists of ℓ 'copies' of the subspace $S_r = \operatorname{span}\{W\} \subset \mathbb{C}^n$.
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Approach 1

[Freund 2005]

- Project the first order system using $\mathcal{V}$

$$\left(\mathcal{V}^H \mathcal{E}_F \mathcal{V}\right) \mathcal{V}^H \frac{d}{dt} z(t) + \left(\mathcal{V}^H \mathcal{A}_F \mathcal{V}\right) \mathcal{V}^H z(t) = \left(\mathcal{V}^H \mathcal{B}_F\right) u(t)$$

$$y(t) = \mathcal{D}_F u(t) + (\mathcal{C}_F \mathcal{V})\, \mathcal{V}^H z(t)$$

with

$$\mathcal{V}^H \mathcal{B}_F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ V^H B \end{bmatrix}, \mathcal{V}^H \mathcal{A}_F \mathcal{V} = \begin{bmatrix} 0 & -I_n & 0 & \cdots & 0 \\ 0 & 0 & -I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -I_n \\ VP_0 V^H & VP_1 V^H & VP_2 V^H & \cdots & VP_{\ell-1} V^H \end{bmatrix},$$

$$\mathcal{V}^H \mathcal{E}_F \mathcal{V} = \begin{bmatrix} I_{(\ell-1)n} & \\ & V^H P_\ell V \end{bmatrix}, \quad \mathcal{C}_F \mathcal{V} = [C_0 V \;\; C_1 V \;\; \cdots \;\; C_{\ell-1} V], \quad \mathcal{D}_F = D.$$

- An ℓth order reduced order system can be read off immediately.
- The first moments of the reduced order system match those of the original system.

Approach 1 [Freund 2005]

- Project the first order system using $\mathcal{V}$

$$\left(\mathcal{V}^H\mathcal{E}_F\mathcal{V}\right)\mathcal{V}^H\frac{d}{dt}z(t)+\left(\mathcal{V}^H\mathcal{A}_F\mathcal{V}\right)\mathcal{V}^Hz(t)=\left(\mathcal{V}^H\mathcal{B}_F\right)u(t)$$

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Approach 1 and 2

- Approach 1 and 2 use companion form linearization.
- Approach 1 uses block-Krylov subspace $\mathcal{K}_s(\mathcal{M}_F, \mathcal{R}_F)$ with $\mathcal{M}_F = (s_0\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{E}_F$ and $\mathcal{R}_F = (s_0\mathcal{E}_F + \mathcal{A}_F)^{-1}\mathcal{B}_F$.
- Approach 2 uses block-Krylov subspace $\mathcal{K}_s(\mathcal{M}_B, \mathcal{R}_B)$ with $\mathcal{M}_B = \mathcal{A}_B^{-1}\mathcal{E}_B$ and $\mathcal{R}_B = \mathcal{A}_B^{-1}\mathcal{B}_B$.
- Neither $\lambda\mathcal{E}_F + \mathcal{A}_F$ nor $\lambda\mathcal{E}_B + \mathcal{A}_B$ is structure-preserving, e.g., $(-\lambda\mathcal{E}_F + \mathcal{A}_F)^T \neq \lambda\mathcal{E}_F + \mathcal{A}_F$ and $(-\lambda\mathcal{E}_B + \mathcal{A}_B)^T \neq \lambda\mathcal{E}_B + \mathcal{A}_B$.
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- Approach 2 uses block-Krylov subspace $\mathcal{K}_s(\mathcal{M}_B, \mathcal{R}_B)$ with $\mathcal{M}_B = \mathcal{A}_B^{-1}\mathcal{E}_B$ and $\mathcal{R}_B = \mathcal{A}_B^{-1}\mathcal{B}_B$.
- Neither $\lambda\mathcal{E}_F + \mathcal{A}_F$ nor $\lambda\mathcal{E}_B + \mathcal{A}_B$ is structure-preserving, e.g., $(-\lambda\mathcal{E}_F + \mathcal{A}_F)^T \neq \lambda\mathcal{E}_F + \mathcal{A}_F$ and $(-\lambda\mathcal{E}_B + \mathcal{A}_B)^T \neq \lambda\mathcal{E}_B + \mathcal{A}_B$.
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Vector space $\mathbb{L}_1(P)$ of linearizations – Motivation

Systematic way to construct linearizations that allow for the preservation of structure and/or are better conditioned than the companion forms.

[Mackey, Mackey, Mehl, Mehrmann, SIMAX 2006] = [4M]

$$P(\lambda)x = \sum_{i=0}^{\ell} \lambda^i P_i x$$

$\Longrightarrow$ linearization of size $\ell n \times \ell n$

$$\underbrace{\left(\lambda \left[\begin{array}{c|cccc} P_\ell & 0 & 0 & \cdots & 0 \\ \hline 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n \end{array}\right] + \left[\begin{array}{cccc|c} P_{\ell-1} & P_{\ell-2} & \cdots & P_1 & P_0 \\ \hline -I_n & 0 & \cdots & 0 & 0 \\ 0 & -I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_n & 0 \end{array}\right] \right)}_{\mathcal{L}_1(\lambda)} \begin{bmatrix} \lambda^{\ell-1}x \\ \lambda^{\ell-2}x \\ \vdots \\ \lambda x \\ x \end{bmatrix} = \begin{bmatrix} P(\lambda)x \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

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Thus

$$\mathcal{L}_1(\lambda)\begin{bmatrix}\lambda^{\ell-1}x\\ \lambda^{\ell-2}x\\ \vdots\\ \lambda x\\ x\end{bmatrix} = \begin{bmatrix}P(\lambda)x\\ 0\\ \vdots\\ 0\end{bmatrix} \iff \mathcal{L}_1(\lambda)\cdot(\Lambda_\ell\otimes I_n)x = e_1\otimes P(\lambda)x$$

as

$$\begin{bmatrix}\lambda^{\ell-1}x\\ \lambda^{\ell-2}x\\ \vdots\\ \lambda x\\ x\end{bmatrix} = \left(\begin{bmatrix}\lambda^{\ell-1}\\ \lambda^{\ell-2}\\ \vdots\\ \lambda\\ 1\end{bmatrix}\otimes I_n\right)x = (\Lambda_\ell\otimes I_n)x \quad \text{and} \quad \begin{bmatrix}P(\lambda)x\\ 0\\ \vdots\\ 0\end{bmatrix} = e_1\otimes P(\lambda)x.$$

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Generalize $\mathcal{L}_1(\lambda) \cdot (\Lambda_\ell \otimes I_n) = e_1 \otimes P(\lambda)$ to

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Matrix Polynomials – (Strong) Linearization

Definition (Linearization)

A pencil $\mathcal{L}(\lambda) = \lambda\mathcal{E} + \mathcal{A}$ with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{kn \times kn}$ is called a linearization of $P(\lambda) \in \Pi_\ell^n$ if there exist unimodular matrix polynomials $E(\lambda)$, $F(\lambda)$ such that

$$E(\lambda)\mathcal{L}(\lambda)F(\lambda) = \left[\begin{array}{c|c} P(\lambda) & 0 \\ \hline 0 & I_{(k-1)n} \end{array}\right]$$

for some $k \in \mathbb{N}$. A matrix polynomial $E(\lambda)$ is unimodular if $\det E(\lambda)$ is a nonzero constant.

Theorem [Lancaster, Psarrakos Report 2005]

For regular polynomials $P(\lambda)$:

- any linearization: the Jordan structure of all finite eigenvalues is preserved.
- strong linearization: the Jordan structure of the eigenvalue ∞ is preserved.

Example

$$\lambda P_1 + P_0 = \lambda \begin{bmatrix} 4 & 5 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \implies \lambda_1 = \frac{1}{4}, \ \lambda_2 = \frac{3}{0} = \infty.$$

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Vector space $\mathbb{L}_1(P)$ of linearizations and Approach 1

- Freund considers

$$\mathcal{E}_F \frac{d}{dt} z(t) + \mathcal{A}_F z(t) = \mathcal{B}_F u(t)$$
$$y(t) = \mathcal{D}_F u(t) + \mathcal{C}_F z(t).$$

- Interpret Freund's approach in terms of the first companion form $\mathcal{L}_1(\lambda) = \lambda\mathcal{E}_1 + \mathcal{A}_1$

$$\mathcal{E}_1 \frac{d}{dt} \widetilde{z}(t) + \mathcal{A}_1 \widetilde{z}(t) = \mathcal{B}_1 u(t)$$
$$y(t) = \mathcal{D}_F u(t) + \mathcal{C}_1 \widetilde{z}(t).$$

with

$$\widetilde{z}(t) = \mathcal{P}^T \widetilde{z}(t)$$
$$\mathcal{B}_1 = \mathcal{P}^T \mathcal{B}$$
$$\mathcal{C}_1 = \mathcal{C}_F \mathcal{P}$$

as $\mathcal{L}_1(\lambda) = \lambda\mathcal{E}_1 + \mathcal{A}_1 = \lambda\mathcal{P}^T\mathcal{E}_F\mathcal{P} + \mathcal{P}^T\mathcal{A}_F\mathcal{P}$ with $\mathcal{P} = \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix}$.

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- Approach is based on the Krylov subspace induced by $\mathcal{M} = (\mathcal{L}_1(s_0))^{-1}\mathcal{E}_1$ and $\mathcal{R} = (\mathcal{L}_1(s_0))^{-1}\mathcal{B}_1$.
- All linearizations in $\mathbb{L}_1$ can be written as

$$\mathcal{L}(\lambda) = [v \otimes I_n \quad W]\mathcal{L}_1(\lambda) = \mathcal{T}\mathcal{L}_1(\lambda) = \lambda\mathcal{T}\mathcal{E}_1 + \mathcal{T}\mathcal{A}_1$$

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- As

$$(\mathcal{T}\mathcal{E}_1)\,\frac{d}{dt}z(t) + (\mathcal{T}\mathcal{A}_1)\,z(t) = (\mathcal{T}\mathcal{B}_1)\,u(t)$$

and

$$(\mathcal{L}(s_0))^{-1}(\mathcal{T}\mathcal{E}_1) = (\mathcal{L}_1(s_0))^{-1}\mathcal{E}_1 = \mathcal{M},$$
$$(\mathcal{L}(s_0))^{-1}(\mathcal{T}\mathcal{B}_1) = (\mathcal{L}_1(s_0))^{-1}\mathcal{B}_1 = \mathcal{R},$$

all linearization in $\mathbb{L}_1$ will yield (theoretically) the same reduced order system.

- A similar observation holds for Approach 2.

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$$\mathcal{L}(\lambda) = [v \otimes I_n \;\; W]\mathcal{L}_1(\lambda) = \mathcal{T}\mathcal{L}_1(\lambda) = \lambda\mathcal{T}\mathcal{E}_1 + \mathcal{T}\mathcal{A}_1$$

with $v \in \mathbb{R}^\ell$, $W \in \mathbb{R}^{\ell n \times (\ell-1)n}$ such that $\mathcal{T} = [v \otimes I_n \;\; W]$ is nonsingular.

- As

$$(\mathcal{T}\mathcal{E}_1)\frac{d}{dt}z(t) + (\mathcal{T}\mathcal{A}_1)z(t) = (\mathcal{T}\mathcal{B}_1)u(t)$$

and

$$(\mathcal{L}(s_0))^{-1}(\mathcal{T}\mathcal{E}_1) = (\mathcal{L}_1(s_0))^{-1}\mathcal{E}_1 = \mathcal{M},$$
$$(\mathcal{L}(s_0))^{-1}(\mathcal{T}\mathcal{B}_1) = (\mathcal{L}_1(s_0))^{-1}\mathcal{B}_1 = \mathcal{R},$$

all linearization in $\mathbb{L}_1$ will yield (theoretically) the same reduced order system.

- A similar observation holds for Approach 2.

Vector space $\mathbb{L}_1(P)$ of linearizations and Approach 1

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- A similar observation holds for Approach 2.

Vector space $\mathbb{L}_1(P)$ – Structured Linearizations

Gyroscopic system $P(\lambda) = P(-\lambda)^T \in \Pi_2^n$

$$P(\lambda) = \lambda^2 M + \lambda G + K, \qquad M = M^T, G = -G^T, K = K^T, \quad M, G, K \in \mathbb{R}^{n\times n}.$$

Companion form in $\mathbb{L}_1(P)$

$$\mathcal{L}_1(\lambda) = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} G & K \\ -I & 0 \end{bmatrix}$$

is not structure preserving as $\mathcal{L}_1(\lambda) \neq \mathcal{L}_1(-\lambda)^T$.

Structured linearization in $\mathbb{L}_1(P)$

$$\mathcal{L}(\lambda) = \lambda \begin{bmatrix} 0 & -M \\ M & G \end{bmatrix} + \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix} \in \mathbb{L}_1(P)$$

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Vector space $\mathbb{L}_1(P)$ – Structured Linearizations

Robot $P(\lambda) = P(-\lambda)^T \in \Pi_4^n$

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0, \qquad P_i = (-1)^i P_i^T, \quad P_i \in \mathbb{R}^{n\times n}, i = 0, \ldots, 4.$$

Companion form in $\mathbb{L}_1(P)$

$$\mathcal{L}_1(\lambda) = \lambda \begin{bmatrix} P_4 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix} + \begin{bmatrix} P_3 & P_2 & P_1 & P_0 \\ -I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \end{bmatrix}$$

Structured linearizations in $\mathbb{L}_1(P)$ different [4M]

$$\mathcal{L}(\lambda) = \lambda \begin{bmatrix} 0 & -P_4 & 0 & -P_4 \\ P_4 & P_3 & P_4 & P_3 \\ 0 & -P_4 & P_1 - P_3 & P_0 - P_2 \\ P_4 & P_3 & P_2 - P_0 & P_1 \end{bmatrix} + \begin{bmatrix} P_4 & 0 & P_4 & 0 \\ 0 & P_2 - P_4 & P_1 - P_3 & P_0 \\ P_4 & P_3 - P_1 & P_2 - P_0 & 0 \\ 0 & P_0 & 0 & P_0 \end{bmatrix}$$

Vector space $\mathbb{L}_1(P)$ – Structured Linearizations

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Companion form in $\mathbb{L}_1(P)$

$$\mathcal{L}_1(\lambda) = \lambda \begin{bmatrix} P_4 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix} + \begin{bmatrix} P_3 & P_2 & P_1 & P_0 \\ -I_n & 0 & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & -I_n & 0 \end{bmatrix}$$

Structured linearizations in $\mathbb{L}_1(P)$

different [4M]

$$\mathcal{L}(\lambda) = \lambda \begin{bmatrix} 0 & -P_4 & 0 & -P_4 \\ P_4 & P_3 & P_4 & P_3 \\ 0 & -P_4 & P_1 - P_3 & P_0 - P_2 \\ P_4 & P_3 & P_2 - P_0 & P_1 \end{bmatrix} + \begin{bmatrix} P_4 & 0 & P_4 & 0 \\ 0 & P_2 - P_4 & P_1 - P_3 & P_0 \\ P_4 & P_3 - P_1 & P_2 - P_0 & 0 \\ 0 & P_0 & 0 & P_0 \end{bmatrix}$$

Vector space $\mathbb{L}_1(P)$ – Structured Linearizations

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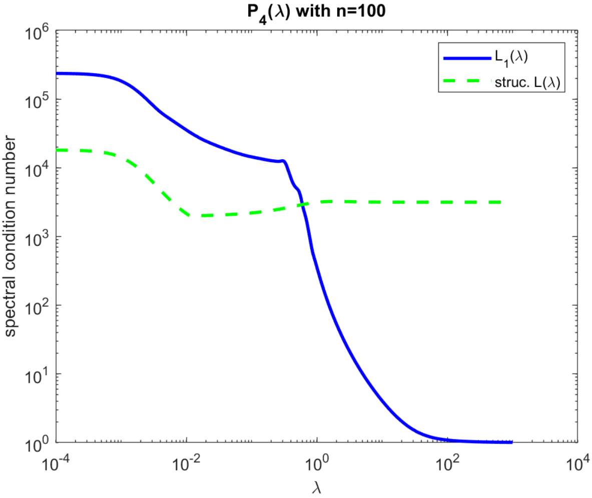
$$\mathcal{L}(\lambda) = \lambda \begin{bmatrix} 0 & -P_4 & 0 & -P_4 \\ P_4 & P_3 & P_4 & P_3 \\ 0 & -P_4 & P_1 - P_3 & P_0 - P_2 \\ P_4 & P_3 & P_2 - P_0 & P_1 \end{bmatrix} + \begin{bmatrix} P_4 & 0 & P_4 & 0 \\ 0 & P_2 - P_4 & P_1 - P_3 & P_0 \\ P_4 & P_3 - P_1 & P_2 - P_0 & 0 \\ 0 & P_0 & 0 & P_0 \end{bmatrix}$$

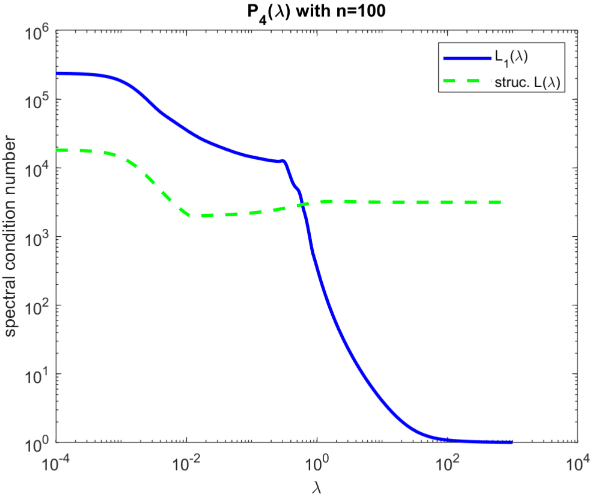
Vector space $\mathbb{L}_1(P)$ – Structured Linearizations



$\mathcal{L}_1(\lambda)$ and $\mathcal{L}(\lambda)$ may be very differently conditioned.

```
P0=1/100*gallery('poisson',10);
P2=randn(100);P2=(P2+P2')/30;
P4=eye(n);
P1=rand(100);P1=P1-P1';
P3=randn(100);P3=P3-P3';
```

Vector space $\mathbb{L}_1(P)$ – Structured Linearizations



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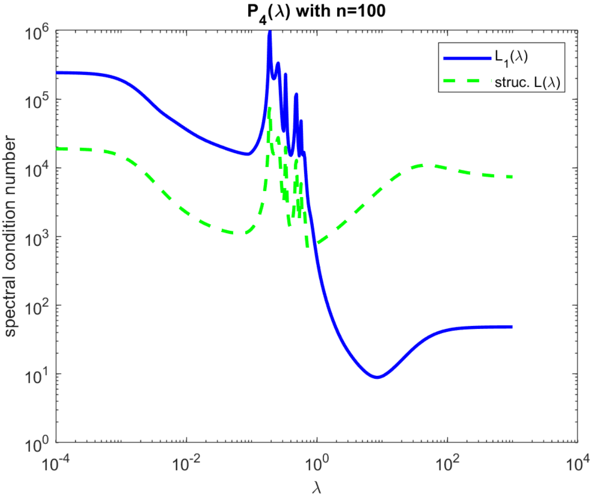
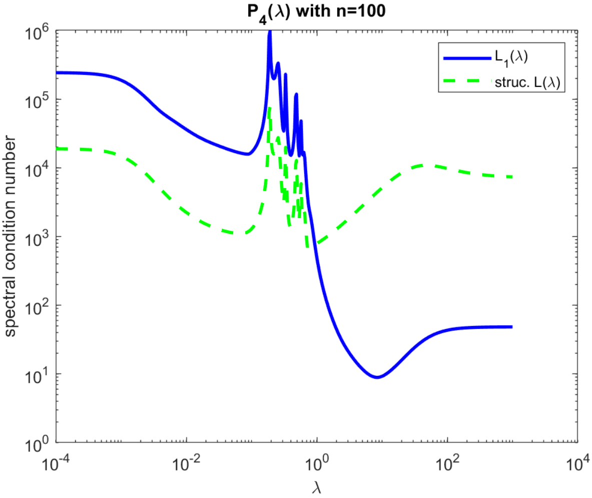


$\mathcal{L}_1(\lambda)$ and $\mathcal{L}(\lambda)$ may be very differently conditioned.

$\mathcal{L}(\lambda)$ is not (block) sparse, while $\mathcal{L}_1(\lambda)$ is.

```
P0=1/100*gallery('poisson',10);
P2=randn(100);P2=(P2+P2')/5;
P4=.5*gallery('poisson',10);
P1=rand(100);P1=P1-P1';
P3=randn(100);P3=P3-P3';
```

Vector space $\mathbb{L}_1(P)$ – Structured Linearizations



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P1=rand(100);P1=P1-P1';
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```

Structured Linearization not in $\mathbb{L}_1(P)$

Robot $P(\lambda) = P(-\lambda)^T \in \Pi_4^n$

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0, \qquad P_i = (-1)^i P_i^T, \quad P_i \in \mathbb{R}^{n\times n}, i = 0, \ldots, 4.$$

(Structured) Linearization not in $\mathbb{L}_1(P)$

$$\mathcal{L}(\lambda) = \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right] = \lambda \mathcal{E} + \mathcal{A} \quad \text{Note: } \mathcal{E}, \mathcal{A} \in \mathbb{R}^{5n\times 5n}!$$

as

$$\mathcal{V}(\lambda)\mathcal{L}(\lambda)\mathcal{U}(\lambda) = \operatorname{diag}(I_{4n}, P(\lambda))$$

for

$$\mathcal{V}(\lambda) = \begin{bmatrix} I_n & 0 & 0 & -P_4 & -\lambda P_4 \\ -\lambda I_n & I_n & 0 & \lambda P_4 & \lambda^2 P_4 + \lambda P_3 + P_2 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & I_n \\ \lambda^2 I_n & -\lambda I_n & I_n & -\lambda^2 P_4 & -\lambda^3 P_4 - \lambda^2 P_3 - \lambda P_2 \end{bmatrix}, \mathcal{U}(\lambda) = \begin{bmatrix} 0 & 0 & I_n & \lambda I_n & \lambda^2 I_n \\ 0 & 0 & 0 & I_n & \lambda I_n \\ 0 & 0 & 0 & 0 & I_n \\ I_n & 0 & 0 & 0 & -\lambda^2 P_4 \\ 0 & I_n & 0 & 0 & \lambda^3 P_4 + \lambda^2 P_3 + \lambda P_2 \end{bmatrix},$$

$\det \mathcal{U}(\lambda) = \det \mathcal{V}(\lambda) = 1.$

Structured Linearization not in $\mathbb{L}_1(P)$

Robot $P(\lambda) = P(-\lambda)^T \in \Pi_4^n$

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0, \qquad P_i = (-1)^i P_i^T, \quad P_i \in \mathbb{R}^{n\times n}, i = 0, \ldots, 4.$$

(Structured) Linearization not in $\mathbb{L}_1(P)$

$$\mathcal{L}(\lambda) = \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right] = \lambda\mathcal{E} + \mathcal{A} \quad \text{Note+}\mathcal{E}, \mathcal{A} \in \mathbb{R}^{5n\times 5n}!$$

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Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

Definition [Block Kronecker Ansatz space] [FS-2]

Let $P(\lambda) \in \Pi_\ell^n$ with $\ell = r + s + 1$. The block Kronecker ansatz space $\mathbb{G}_{r+1}(P)$ is the set of all $\ell n \times \ell n$ matrix pencils $\mathbb{L}(\lambda)$ that satisfy the block Kronecker ansatz equation

$$\left[\begin{array}{c|c} [\lambda^r I_n \ \cdots \ I_n] & \\ \hline & I_{sn} \end{array}\right] \overbrace{\left[\begin{array}{c|c} \mathcal{L}_{11}(\lambda) & \mathcal{L}_{12}(\lambda) \\ \hline \mathcal{L}_{21}(\lambda) & \mathcal{L}_{22}(\lambda) \end{array}\right]}^{\mathcal{L}(\lambda)} \left[\begin{array}{c|c} \begin{bmatrix} \lambda^s I_n \\ \vdots \\ I_n \end{bmatrix} & \\ \hline & I_{rn} \end{array}\right] = \left[\begin{array}{c|c} \alpha P(\lambda) & 0 \\ \hline 0 & 0 \end{array}\right].$$

- $\mathbb{G}_{r+1}(P)$ is a vector space over $\mathbb{R}$ of dimension $(\ell - 1)\ell n^2 + 1$. [FS-2]
- Thus, $\mathbb{L}_1(P) \neq \mathbb{G}_{r+1}(P)$.
- Almost all pencils in $\mathbb{G}_{r+1}(P)$ are strong linearizations of $P(\lambda)$. [FS-2]

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Higher order system and block Kronecker linearizations

Robot $P(\lambda) \in \Pi_4^n$

$$P_4 \frac{d^4}{dt^4}x(t) + P_3 \frac{d^3}{dt^3}x(t) + P_2 \frac{d^2}{dt^2}x(t) + P_1 \frac{d}{dt}x(t) + P_0 x(t) = Bu(t)$$

$$Du(t) + C_3 \frac{d^3}{dt^3}x(t) + C_2 \frac{d^2}{dt^2}x(t) + C_1 \frac{d}{dt}x(t) + C_0 x(t) = y(t)$$

The linearization

$$\mathcal{L}(\lambda) = \lambda\mathcal{E} + \mathcal{A} = \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right]$$

does not give an equivalent first order ODE of the form $\mathcal{E}\frac{d}{dt}z(t) + \mathcal{A}z(t) = \mathcal{B}u(t)$

as $\begin{bmatrix} \lambda^2 I_n & -\lambda I_n & I_n & 0 & 0 \end{bmatrix} \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right] \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ I_n \\ 0 \\ 0 \end{bmatrix} = P(\lambda).$

Higher order system and block Kronecker linearizations

Robot $P(\lambda) \in \Pi_4^n$

$$P_4 \frac{d^4}{dt^4} x(t) + P_3 \frac{d^3}{dt^3} x(t) + P_2 \frac{d^2}{dt^2} x(t) + P_1 \frac{d}{dt} x(t) + P_0 x(t) = Bu(t)$$
$$Du(t) + C_3 \frac{d^3}{dt^3} x(t) + C_2 \frac{d^2}{dt^2} x(t) + C_1 \frac{d}{dt} x(t) + C_0 x(t) = y(t)$$

The linearization

$$\mathcal{L}(\lambda) = \lambda\mathcal{E} + \mathcal{A} = \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right]$$

does not give an equivalent first order ODE of the form $\mathcal{E}\frac{d}{dt}z(t) + \mathcal{A}z(t) = \mathcal{B}u(t)$

as $\begin{bmatrix} \lambda^2 I_n & -\lambda I_n & I_n & 0 & 0 \end{bmatrix} \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right] \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ I_n \\ 0 \\ 0 \end{bmatrix} = P(\lambda).$

Higher order system and block Kronecker linearizations

Robot $P(\lambda) \in \Pi_4^n$

$$P_4 \frac{d^4}{dt^4}x(t) + P_3 \frac{d^3}{dt^3}x(t) + P_2 \frac{d^2}{dt^2}x(t) + P_1 \frac{d}{dt}x(t) + P_0 x(t) = Bu(t)$$

$$Du(t) + C_3 \frac{d^3}{dt^3}x(t) + C_2 \frac{d^2}{dt^2}x(t) + C_1 \frac{d}{dt}x(t) + C_0 x(t) = y(t)$$

The linearization

$$\mathcal{L}(\lambda) = \lambda\mathcal{E} + \mathcal{A} = \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right]$$

does not give an equivalent first order ODE of the form $\mathcal{E}\frac{d}{dt}z(t) + \mathcal{A}z(t) = \mathcal{B}u(t)$

as $\begin{bmatrix} \lambda^2 I_n & -\lambda I_n & I_n & 0 & 0 \end{bmatrix} \left[\begin{array}{ccc|cc} P_4 & 0 & 0 & I & 0 \\ 0 & -P_2 - \lambda P_3 & 0 & \lambda I & I \\ 0 & 0 & P_0 + \lambda P_1 & 0 & \lambda I \\ \hline I & -\lambda I & 0 & 0 & 0 \\ 0 & I & -\lambda I & 0 & 0 \end{array}\right] \begin{bmatrix} \lambda^2 I_n \\ \lambda I_n \\ I_n \\ 0 \\ 0 \end{bmatrix} = P(\lambda).$

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

In $\mathbb{L}_1$ all linearizations are based on $\mathcal{L}_1(\lambda)$, the linearizations in $\mathbb{G}_{r+1}$ are based on

$$\mathcal{L}_K(\lambda) = \lambda\mathcal{E}_K + \mathcal{A}_K$$

$$= \left[\begin{array}{cccc|cccc} \lambda\alpha P_\ell + \alpha P_{\ell-1} & \alpha P_{\ell-2} & \cdots & \alpha P_r & -I_n & & & \\ & & & \alpha P_{r-1} & \lambda I_n & \ddots & & \\ & & & \vdots & & \ddots & -I_n & \\ & & & \alpha P_0 & & & \lambda I_n & \\ \hline -I_n & \lambda I_n & & & & & & \\ & \ddots & \ddots & & & 0 & & \\ & & -I_n & \lambda I_n & & & & \end{array}\right]$$

$$= \begin{bmatrix} \Sigma_r(\lambda) & L_r^T(\lambda) \\ L_s(\lambda) & 0 \end{bmatrix}$$

with $\ell = r + s + 1$, $\Sigma_r(\lambda) \in \mathbb{C}^{(r+1)n \times sn}$, and $L_j(\lambda) \in \mathbb{C}^{jn \times (j+1)n}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- We can find $\mathcal{B}_K, \mathcal{C}_K$ such that

$$G(s) = D + \sum_{j=0}^{\ell-1} C_j((P(s))^{-1}B = \mathcal{D}_K + \mathcal{C}_K\left(\mathcal{L}_K(s)\right)^{-1}\mathcal{B}_K.$$

- Introduce shift $s_0 \in \mathbb{C}$ such that $\mathcal{L}_K(s_0) = s_0\mathcal{E}_K + \mathcal{A}_K$ is nonsingular. Then

$$G(s) = \mathcal{D}_K + \mathcal{C}_K(\mathcal{L}_K(s))^{-1}\mathcal{B}_K = \mathcal{D}_K + \mathcal{C}_K(I + (s - s_0)\mathcal{M}_K)^{-1}\mathcal{R}_K$$

with

$$\mathcal{M}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{E}_K, \qquad \mathcal{R}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{B}_K.$$

- Compute basis of $\mathcal{K}_s(\mathcal{M}_K, \mathcal{R}_K)$. Represent the basis in block form

$$\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_\ell \end{bmatrix}, \qquad W_j \in \mathbb{C}^{n\times r}.$$

- Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- We can find $\mathcal{B}_K, \mathcal{C}_K$ such that

$$G(s) = D + \sum_{j=0}^{\ell-1} C_j((P(s))^{-1}B = \mathcal{D}_K + \mathcal{C}_K\left(\mathcal{L}_K(s)\right)^{-1}\mathcal{B}_K.$$

- Introduce shift $s_0 \in \mathbb{C}$ such that $\mathcal{L}_K(s_0) = s_0\mathcal{E}_K + \mathcal{A}_K$ is nonsingular. Then

$$G(s) = \mathcal{D}_K + \mathcal{C}_K(\mathcal{L}_K(s))^{-1}\mathcal{B}_K = \mathcal{D}_K + \mathcal{C}_K(I + (s - s_0)\mathcal{M}_K)^{-1}\mathcal{R}_K$$

with

$$\mathcal{M}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{E}_K, \qquad \mathcal{R}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{B}_K.$$

- Compute basis of $\mathcal{K}_s(\mathcal{M}_K, \mathcal{R}_K)$. Represent the basis in block form

$$\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_\ell \end{bmatrix}, \qquad W_j \in \mathbb{C}^{n\times r}.$$

- Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- We can find $\mathcal{B}_K, \mathcal{C}_K$ such that

$$G(s) = D + \sum_{j=0}^{\ell-1} C_j((P(s))^{-1}B = \mathcal{D}_K + \mathcal{C}_K\left(\mathcal{L}_K(s)\right)^{-1}\mathcal{B}_K.$$

- Introduce shift $s_0 \in \mathbb{C}$ such that $\mathcal{L}_K(s_0) = s_0\mathcal{E}_K + \mathcal{A}_K$ is nonsingular. Then

$$G(s) = \mathcal{D}_K + \mathcal{C}_K(\mathcal{L}_K(s))^{-1}\mathcal{B}_K = \mathcal{D}_K + \mathcal{C}_K(I + (s - s_0)\mathcal{M}_K)^{-1}\mathcal{R}_K$$

with

$$\mathcal{M}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{E}_K, \qquad \mathcal{R}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{B}_K.$$

- Compute basis of $\mathcal{K}_s(\mathcal{M}_K, \mathcal{R}_K)$. Represent the basis in block form

$$\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_\ell \end{bmatrix}, \qquad W_j \in \mathbb{C}^{n\times r}.$$

- Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- We can find $\mathcal{B}_K, \mathcal{C}_K$ such that

$$G(s) = D + \sum_{j=0}^{\ell-1} C_j((P(s))^{-1}B = \mathcal{D}_K + \mathcal{C}_K\,(\mathcal{L}_K(s))^{-1}\,\mathcal{B}_K.$$

- Introduce shift $s_0 \in \mathbb{C}$ such that $\mathcal{L}_K(s_0) = s_0\mathcal{E}_K + \mathcal{A}_K$ is nonsingular. Then

$$G(s) = \mathcal{D}_K + \mathcal{C}_K(\mathcal{L}_K(s))^{-1}\mathcal{B}_K = \mathcal{D}_K + \mathcal{C}_K(I + (s - s_0)\mathcal{M}_K)^{-1}\mathcal{R}_K$$

with

$$\mathcal{M}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{E}_K, \qquad \mathcal{R}_K = (\mathcal{L}_K(s_0))^{-1}\mathcal{B}_K.$$

- Compute basis of $\mathcal{K}_s(\mathcal{M}_K, \mathcal{R}_K)$. Represent the basis in block form

$$\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_\ell \end{bmatrix}, \qquad W_j \in \mathbb{C}^{n\times r}.$$

- Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- Any linearization in $\mathbb{G}_{r+1}$ can be expressed as

$$\widetilde{\mathcal{L}}_K(\lambda) = \mathcal{T}_1 \mathcal{L}_K(\lambda) \mathcal{T}_2 \quad \text{with } \mathcal{T}_1 = \left[\begin{array}{c|c} I_{(r+1)n} & B_1 \\ \hline 0 & C_1 \end{array}\right], \quad \mathcal{T}_2 = \left[\begin{array}{c|c} I_{(s+1)n} & 0 \\ \hline B_2 & C_2 \end{array}\right]$$

and $B_1 \in \mathbb{R}^{(r+1)n \times sn}$, $B_2 \in \mathbb{R}^{rn \times (s+1)n}$, $C_1 \in \mathbb{R}^{sn \times sn}$, $C_2 \in \mathbb{R}^{rn \times rn}$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K(\widetilde{\mathcal{L}}_K(s))^{-1}\widetilde{\mathcal{B}}_K$ with $\widetilde{\mathcal{C}}_K = \mathcal{C}_K\mathcal{T}_2$, $\widetilde{\mathcal{B}}_K = \mathcal{T}_1\mathcal{B}_K$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K(I + (s - s_0)\widetilde{\mathcal{M}}_K)^{-1}\widetilde{\mathcal{R}}_K$ with

$$\begin{aligned} \widetilde{\mathcal{M}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1}\mathcal{T}_1\mathcal{E}_K\mathcal{T}_2, & \widetilde{\mathcal{R}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1}\widetilde{\mathcal{B}}_K, \\ &= \mathcal{T}_2^{-1}\mathcal{M}_K\mathcal{T}_2, & &= \mathcal{T}_2^{-1}\mathcal{R}_K. \end{aligned}$$

- Thus, $\mathcal{K}(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_k) = \mathcal{T}_2^{-1}\mathcal{K}(\mathcal{M}_K, \mathcal{R}_k)$.

- As before: Compute basis of $\mathcal{K}_s(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_K)$. Represent it in block form with blocks $W_j \in \mathbb{C}^{n \times r}$, $j = 1, \ldots, \ell$. Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- Any linearization in $\mathbb{G}_{r+1}$ can be expressed as

$$\widetilde{\mathcal{L}}_K(\lambda) = \mathcal{T}_1 \mathcal{L}_K(\lambda) \mathcal{T}_2 \quad \text{with} \;\; \mathcal{T}_1 = \left[\begin{array}{c|c} I_{(r+1)n} & B_1 \\ \hline 0 & C_1 \end{array}\right], \quad \mathcal{T}_2 = \left[\begin{array}{c|c} I_{(s+1)n} & 0 \\ \hline B_2 & C_2 \end{array}\right]$$

and $B_1 \in \mathbb{R}^{(r+1)n \times sn}$, $B_2 \in \mathbb{R}^{rn \times (s+1)n}$, $C_1 \in \mathbb{R}^{sn \times sn}$, $C_2 \in \mathbb{R}^{rn \times rn}$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K(\widetilde{\mathcal{L}}_K(s))^{-1}\widetilde{\mathcal{B}}_K$ with $\widetilde{\mathcal{C}}_K = \mathcal{C}_K \mathcal{T}_2$, $\widetilde{\mathcal{B}}_K = \mathcal{T}_1 \mathcal{B}_K$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K(I + (s - s_0)\widetilde{\mathcal{M}}_K)^{-1}\widetilde{\mathcal{R}}_K$ with

$$\widetilde{\mathcal{M}}_K = (\widetilde{\mathcal{L}}_K(s_0))^{-1}\mathcal{T}_1\mathcal{E}_K\mathcal{T}_2, \qquad \widetilde{\mathcal{R}}_K = (\widetilde{\mathcal{L}}_K(s_0))^{-1}\widetilde{\mathcal{B}}_K,$$
$$= \mathcal{T}_2^{-1}\mathcal{M}_K\mathcal{T}_2, \qquad = \mathcal{T}_2^{-1}\mathcal{R}_K.$$

- Thus, $\mathcal{K}(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_k) = \mathcal{T}_2^{-1}\mathcal{K}(\mathcal{M}_K, \mathcal{R}_k)$.

- As before: Compute basis of $\mathcal{K}_s(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_K)$. Represent it in block form with blocks $W_j \in \mathbb{C}^{n \times r}$, $j = 1, \ldots, \ell$. Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- Any linearization in $\mathbb{G}_{r+1}$ can be expressed as

$$\widetilde{\mathcal{L}}_K(\lambda) = \mathcal{T}_1 \mathcal{L}_K(\lambda) \mathcal{T}_2 \quad \text{with } \mathcal{T}_1 = \left[\begin{array}{c|c} I_{(r+1)n} & B_1 \\ \hline 0 & C_1 \end{array}\right], \quad \mathcal{T}_2 = \left[\begin{array}{c|c} I_{(s+1)n} & 0 \\ \hline B_2 & C_2 \end{array}\right]$$

  and $B_1 \in \mathbb{R}^{(r+1)n \times sn}$, $B_2 \in \mathbb{R}^{rn \times (s+1)n}$, $C_1 \in \mathbb{R}^{sn \times sn}$, $C_2 \in \mathbb{R}^{rn \times rn}$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K (\widetilde{\mathcal{L}}_K(s))^{-1} \widetilde{\mathcal{B}}_K$ with $\widetilde{\mathcal{C}}_K = \mathcal{C}_K \mathcal{T}_2$, $\widetilde{\mathcal{B}}_K = \mathcal{T}_1 \mathcal{B}_K$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K (I + (s - s_0)\widetilde{\mathcal{M}}_K)^{-1} \widetilde{\mathcal{R}}_K$ with

$$\begin{aligned} \widetilde{\mathcal{M}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1} \mathcal{T}_1 \mathcal{E}_K \mathcal{T}_2, & \widetilde{\mathcal{R}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1} \widetilde{\mathcal{B}}_K, \\ &= \mathcal{T}_2^{-1} \mathcal{M}_K \mathcal{T}_2, & &= \mathcal{T}_2^{-1} \mathcal{R}_K. \end{aligned}$$

- Thus, $\mathcal{K}(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_k) = \mathcal{T}_2^{-1} \mathcal{K}(\mathcal{M}_K, \mathcal{R}_k)$.

- As before: Compute basis of $\mathcal{K}_s(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_K)$. Represent it in block form with blocks $W_j \in \mathbb{C}^{n \times r}$, $j = 1, \ldots, \ell$. Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- Any linearization in $\mathbb{G}_{r+1}$ can be expressed as
$$\widetilde{\mathcal{L}}_K(\lambda) = \mathcal{T}_1 \mathcal{L}_K(\lambda) \mathcal{T}_2 \quad \text{with } \mathcal{T}_1 = \left[\begin{array}{c|c} I_{(r+1)n} & B_1 \\ \hline 0 & C_1 \end{array}\right], \quad \mathcal{T}_2 = \left[\begin{array}{c|c} I_{(s+1)n} & 0 \\ \hline B_2 & C_2 \end{array}\right]$$
and $B_1 \in \mathbb{R}^{(r+1)n \times sn}$, $B_2 \in \mathbb{R}^{rn \times (s+1)n}$, $C_1 \in \mathbb{R}^{sn \times sn}$, $C_2 \in \mathbb{R}^{rn \times rn}$.
- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K(\widetilde{\mathcal{L}}_K(s))^{-1}\widetilde{\mathcal{B}}_K$ with $\widetilde{\mathcal{C}}_K = \mathcal{C}_K \mathcal{T}_2$, $\widetilde{\mathcal{B}}_K = \mathcal{T}_1 \mathcal{B}_K$.
- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K(I + (s - s_0)\widetilde{\mathcal{M}}_K)^{-1}\widetilde{\mathcal{R}}_K$ with
$$\begin{aligned} \widetilde{\mathcal{M}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1}\mathcal{T}_1 \mathcal{E}_K \mathcal{T}_2, & \widetilde{\mathcal{R}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1}\widetilde{\mathcal{B}}_K, \\ &= \mathcal{T}_2^{-1}\mathcal{M}_K \mathcal{T}_2, & &= \mathcal{T}_2^{-1}\mathcal{R}_K. \end{aligned}$$
- Thus, $\mathcal{K}(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_k) = \mathcal{T}_2^{-1}\mathcal{K}(\mathcal{M}_K, \mathcal{R}_k)$.
- As before: Compute basis of $\mathcal{K}_s(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_K)$. Represent it in block form with blocks $W_j \in \mathbb{C}^{n \times r}$, $j = 1, \ldots, \ell$. Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Block Kronecker Ansatz space $\mathbb{G}_{r+1}$

- Any linearization in $\mathbb{G}_{r+1}$ can be expressed as

$$\widetilde{\mathcal{L}}_K(\lambda) = \mathcal{T}_1 \mathcal{L}_K(\lambda) \mathcal{T}_2 \quad \text{with } \mathcal{T}_1 = \left[\begin{array}{c|c} I_{(r+1)n} & B_1 \\ \hline 0 & C_1 \end{array}\right], \quad \mathcal{T}_2 = \left[\begin{array}{c|c} I_{(s+1)n} & 0 \\ \hline B_2 & C_2 \end{array}\right]$$

  and $B_1 \in \mathbb{R}^{(r+1)n \times sn}$, $B_2 \in \mathbb{R}^{rn \times (s+1)n}$, $C_1 \in \mathbb{R}^{sn \times sn}$, $C_2 \in \mathbb{R}^{rn \times rn}$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K (\widetilde{\mathcal{L}}_K(s))^{-1} \widetilde{\mathcal{B}}_K$ with $\widetilde{\mathcal{C}}_K = \mathcal{C}_K \mathcal{T}_2$, $\widetilde{\mathcal{B}}_K = \mathcal{T}_1 \mathcal{B}_K$.

- $G(s) = \mathcal{D}_K + \widetilde{\mathcal{C}}_K (I + (s - s_0) \widetilde{\mathcal{M}}_K)^{-1} \widetilde{\mathcal{R}}_K$ with

$$\begin{aligned} \widetilde{\mathcal{M}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1} \mathcal{T}_1 \mathcal{E}_K \mathcal{T}_2, & \widetilde{\mathcal{R}}_K &= (\widetilde{\mathcal{L}}_K(s_0))^{-1} \widetilde{\mathcal{B}}_K, \\ &= \mathcal{T}_2^{-1} \mathcal{M}_K \mathcal{T}_2, & &= \mathcal{T}_2^{-1} \mathcal{R}_K. \end{aligned}$$

- Thus, $\mathcal{K}(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_k) = \mathcal{T}_2^{-1} \mathcal{K}(\mathcal{M}_K, \mathcal{R}_k)$.

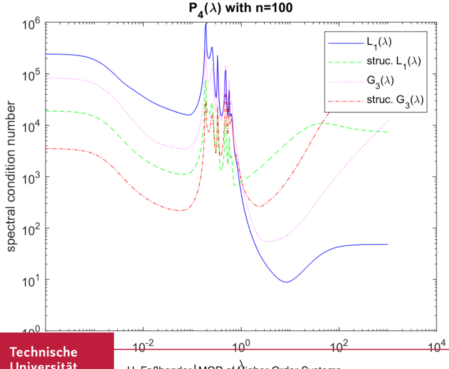
- As before: Compute basis of $\mathcal{K}_s(\widetilde{\mathcal{M}}_K, \widetilde{\mathcal{R}}_K)$. Represent it in block form with blocks $W_j \in \mathbb{C}^{n \times r}$, $j = 1, \ldots, \ell$. Generate reduced order higher order system via projection with V, the matrix representing an orthonormal basis of span$\{W_{r+1}\}$.

Four different Linearizations for Robot Example

Robot $P(\lambda) \in \Pi_4^n$

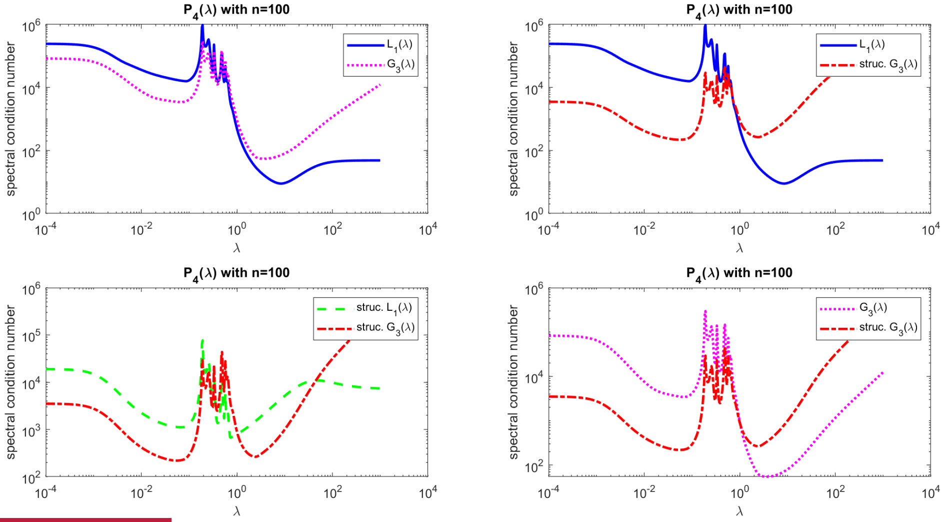
$$P_4 \frac{d^4}{dt^4}x(t) + P_3 \frac{d^3}{dt^3}x(t) + P_2 \frac{d^2}{dt^2}x(t) + P_1 \frac{d}{dt}x(t) + P_0 x(t) = Bu(t), \qquad P_i = (-1)^i P_i^T$$

$$Du(t) + C_3 \frac{d^3}{dt^3}x(t) + C_2 \frac{d^2}{dt^2}x(t) + C_1 \frac{d}{dt}x(t) + C_0 x(t) = y(t)$$




```
P0=1/100*gallery('poisson',10);
P2=randn(100);P2=(P2+P2')/5;
P4=.5*gallery('poisson',10);
P1=rand(100);P1=P1-P1';
P3=randn(100);P3=P3-P3';
```

Four different Linearizations for Robot Example

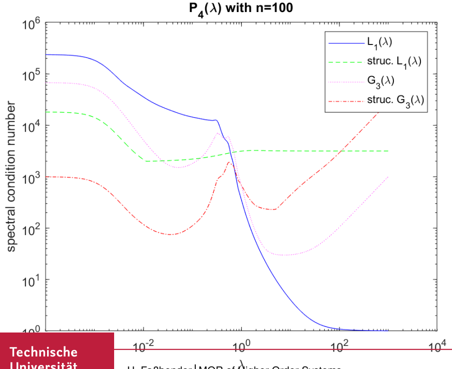


Four different Linearizations for Robot Example

Robot $P(\lambda) \in \Pi_4^n$

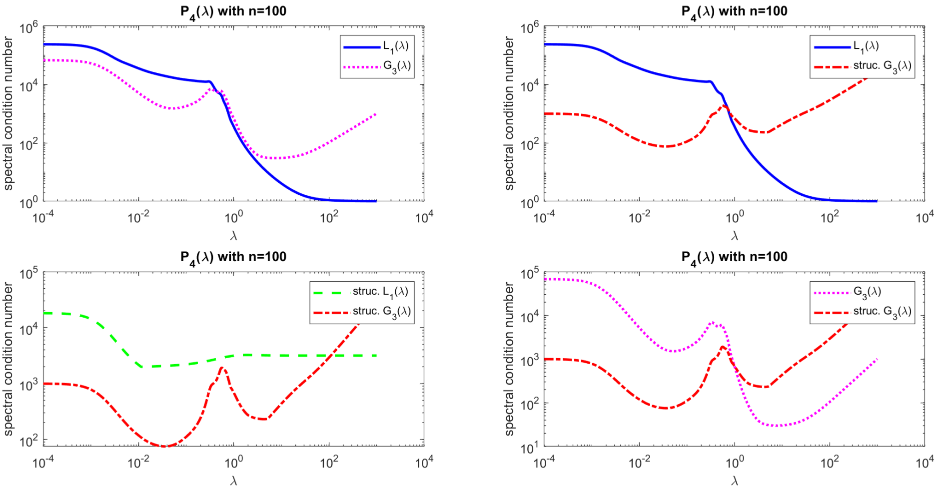
$$P_4 \frac{d^4}{dt^4} x(t) + P_3 \frac{d^3}{dt^3} x(t) + P_2 \frac{d^2}{dt^2} x(t) + P_1 \frac{d}{dt} x(t) + P_0 x(t) = Bu(t), \quad P_i = (-1)^i P_i^T$$

$$Du(t) + C_3 \frac{d^3}{dt^3} x(t) + C_2 \frac{d^2}{dt^2} x(t) + C_1 \frac{d}{dt} x(t) + C_0 x(t) = y(t)$$

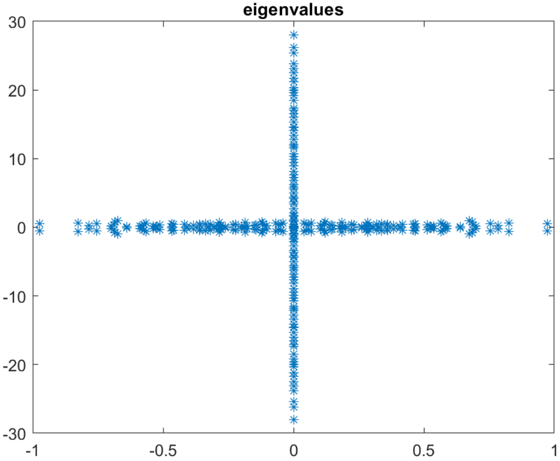



```
P0=1/100*gallery('poisson',10);
P2=randn(100);P2=(P2+P2')/30;
P4=eye(n);
P1=rand(100);P1=P1-P1';
P3=randn(100);P3=P3-P3';
```

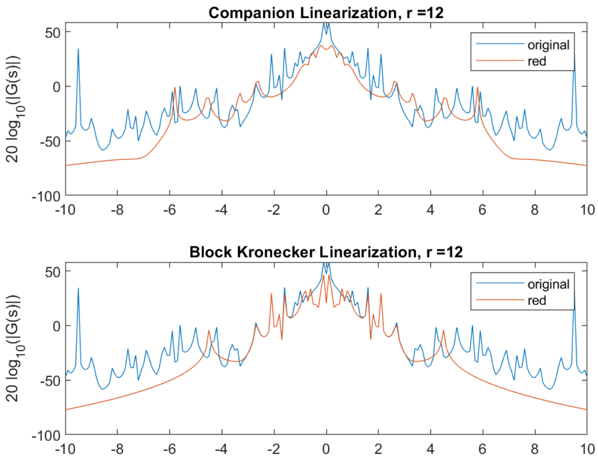
Four different Linearizations for Robot Example



Eigenvalues of Robot Example



MOR for Robot Example, expansion points $\pm 0.5\imath$



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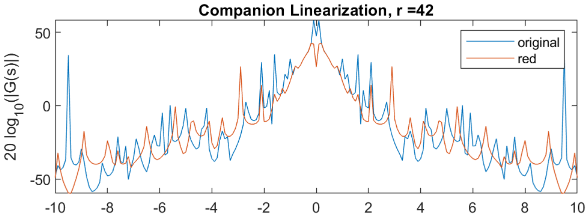

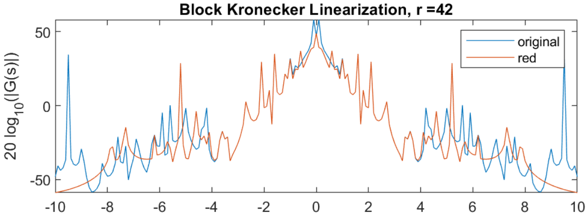

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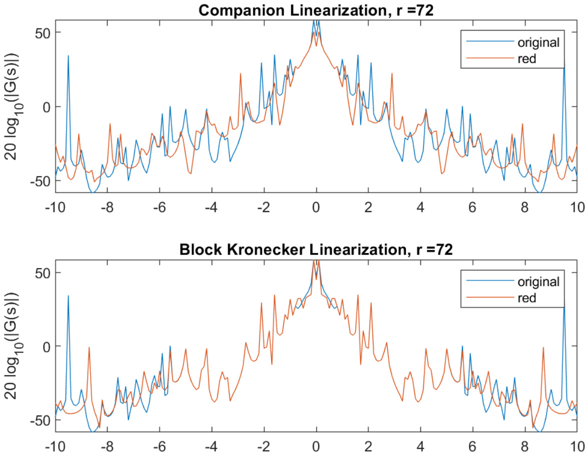

Conclusions

- Galerkin projection based MOR for higher order LTI systems.
- Compute projection from linearization of higher order LTI system such that higher order system can be recovered.
- Vector spaces $\mathbb{L}_1(P)$ and $\mathbb{G}_{r+1}(P)$ allow to generate an abundance of linearizations.
- Linearizations have different condition.
  - It is not (yet) clear how to choose an optimally conditioned linearization.
  - For the structured robot example, the structured linearizations seem to be better conditioned.
- *LU* decomposition of linearization needs to be computed efficiently.
  - For block-dense linearizations, the LU decomposition can be computed in about $\mathcal{O}(\ell^3 n^3)$ flops.
  - For the structured robot example, the LU decomposition of the structured block Kronecker linearization can be computed in just $\mathcal{O}(n^3 + \ell^2 n^2)$ flops.
- Open question: What are the dominant poles of a higher order system?

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Main References

[FS-1]: H. Faßbender and P. Saltenberger, *On vector spaces of linearizations for matrix polynomials in orthogonal bases*. Linear Algebra and its Applications 525 (2017), pp. 59–83.

[FS-2]: H. Faßbender and P. Saltenberger, *Block Kronecker Ansatz Spaces for Matrix Polynomials*. Linear Algebra and its Applications 542 (2018), pp. 118–148.

[Freund BIT 2005]: R. Freund, *Krylov subspaces associated with higher-order linear dynamical systems*. BIT 45 (2005), pp. 495–526.

[4M]: D. S. Mackey, N. Mackey, C. Mehl and V. Mehrmann, *Vector spaces of linearizations for matrix polynomials*. SIAM J. Matrix Anal. Appl., 28 (2006), pp. 971–1004.

[LBLW 2011]: B. Li, L. Bao, Y. Lin, Y. Wei, *Model-order reduction of kth order MIMO dynamical systems using block kth order Krylov subspaces*. International Journal of Computer Mathematics 88(1) (2011), pp. 150–162.

Some results from [FS-2] have been discovered independently in
M. Bueno, F. Dopico, J. Pérez, R. Saavedra, B. Zykoski, *A unified approach to Fiedler-like pencils via strong block minimal bases pencils*. arXiv preprint, arXiv:1611.07170v1.

More References

Dopico, Lawrence, Pérez, Van Dooren, *Block Kronecker linearizations of matrix polynomials and their backward errors*. MIMS-eprint 2016.34.

Freund, *Pade-type model reduction of second-order and higher-order linear dynamical systems*. In Benner, Mehrmann, Sorensen, *Dimensions reduction of large-scale systems*, Springer 2005.

Lancaster, Psarrakos, *A Note on Weak and Strong Linearizations of Regular Matrix Polynomials*, Numerical Analysis Report No. 470, 2005.

Lin, Bao, Wei, *Model-order reduction of large-scale kth order linear dynamical systems via a kth order Arnoldi method*. International Journal of Computer Mathematics 87(2) (2010), pp. 435–453. Mackey, Mackey, Mehl, Mehrmann, *Structured polynomial eigenvalue problems: good vibrations from good linearizations*. SIMAX 28 (2006).

Mehrmann, Schröder, Simoncini, *An implicitly-restarted Krylov subspace method for real symmetric/skew-symmetric eigenproblems*, LAA, 2009.

Mehrmann, Watkins, *Structure-preserving methods for computing eigenpairs of large sparse skew-Hamiltonian/Hamiltonian pencils*, SISC 22 (2001).